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On the admissibility of topological vector spaces

O. HADŽIĆ

1. Introduction. Let X be a Hausdorff topological vector space. A subset A of X is called *admissible* [7] if for every compact subset $K \subset A$ and for every neighbourhood U of zero in X there is some continuous mapping $h\colon K \to A$ such that

(i) $\dim(\operatorname{span} h(K)) < \infty$,

(ii) $x - hx \in U$, for all $x \in K$.

S. HAHN and K. F. PÖTTER [3] proved fixed point theorems for admissible subsets of Hausdorff topological vector spaces. NAGUMO proved that all convex subsets of a locally convex space are admissible [9] and the admissibility of many non-convex topological vector spaces has been proved by KLEE [6], RIEDRICH [14], [15], ICHII [4], PALLASCHKE [12] and KRAUTHAUSEN [7].

But the following questions remained open:

a) Which Hausdorff topological vector spaces are admissible?

b) Which convex subsets are admissible?

c) For which compact subsets K of a Hausdorff topological vector space X is the following valid:

($*$) If U is an arbitrary neighbourhood of zero in X then there is a finite dimensional continuous mapping $h\colon K \to \operatorname{co} K$ such that $x - hx \in U$ for all $x \in K$.

Recently, MATUSOV [8] proved that every compact convex subset of a Hausdorff topological vector space has the fixed point property using an idea of SARIMSAKOV [10] and a result of KASAHARA [5].

Now we give Kasahara's definition of paranormed spaces [5].

A linear mapping Φ of a topological semifield E into another F is said to be *positive* if $\Phi(x) \geqq 0$ in F for every $x \in E$ with $x \geqq 0$. Let $\| \ \|$ be a mapping of a

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linear space X into a topological semifield E and let Φ be a continuous positive linear mapping of E into itself. The triple $(X, \| \|, \Phi)$ is called a *paranormed space over* E and $\| \|$ a *Φ-paranorm* on X over E if the following conditions are satisfied:
(P1) $\|x\| \geqq 0$, for every $x \in X$;
(P2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for every real λ and every $x \in X$;
(P3) $\|x+y\| \leqq \Phi(\|x\| + \|y\|)$ for every $x, y \in X$.

A set $K, K \subset X$ where X is a topological vector space, is said to be of type Φ iff $(X, \| \|, \Phi)$ is a paranormed space and for every $n \in N$, every $x_1, x_2, \ldots, x_n \in K-K$ and every $\lambda_i, 0 \leqq \lambda_i \leqq 1$ $(i=1, 2, \ldots, n)$ such that $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$, we have $\left\| \sum_{i=1}^{n} \lambda_i x_i \right\| \leqq \sum_{i=1}^{n} \lambda_i \Phi(\|x_i\|)$. If $K=X$, the space X is of type Φ.

In this paper we shall prove:

a′) Every Hausdorff topological vector space of type Φ is admissible.

b′) Every convex subset of type Φ of a Hausdorff topological vector space is admissible.

c′) For every compact subset K of type Φ of a Hausdorff topological vector space property $(*)$ is valid.

As a Corollary we shall obtain an extension of Matusov's fixed point theorem.

2. The main result. We use the following theorem from KASAHARA's paper [5].

Let (X, τ) be a topological linear space. Then there exists a paranormed space $(X, \| \|, \Phi)$ over a Tihonov semifield E such that:

(1) *For every neighbourhood U of $0 \in X$ there are an $\varepsilon > 0$ and an indecomposable idempotent $\varrho \in E$ such that*

$$\{x \in X \colon \|x\| \cdot \varrho \leqq \varepsilon \varrho\} \subset U.$$

(2) *For every neighbourhood U of $0 \in E$ the set*

$$\{x \in X \colon \|x\| \in U\}$$

is a neighbourhood of $0 \in X$.

The Tihonov semifield E from the above Theorem is R_Δ, the set of all mappings from Δ into R where Δ is a set of paranorms generating the topology of X and satisfying the condition that for each $p \in \Delta$ there are $\alpha > 0$ and $q \in \Delta$ such that

$$p(x+y) \leqq \alpha\big(q(x) + q(y)\big), \quad \text{for all} \quad x, y \in X.$$

Now we are ready to formulate our main theorem.

Theorem. *For every compact subset K of type Φ of a topological vector space X and for every neighbourhood U of zero in X there exists a finite dimensional continuous mapping h: $K \to \operatorname{co} K$ such that $x - hx \in U$ for all $x \in K$.*

Proof. Let U be an arbitrary neighbourhood of zero in X and let $\mu = \{t_1, t_2, \ldots, t_n\} \subset \Delta$ and $\varepsilon > 0$ such that

$$\|x-y\| \in U_{\mu,\varepsilon} \Rightarrow x-y \in U,$$

where

$$U_{\mu,\varepsilon} = \{u : u \in R_\Delta,\ u(t_j) < \varepsilon,\ j = 1, 2, \ldots, n\}.$$

Further, since the mapping Φ: $R_\Delta \to R_\Delta$ is a continuous linear mapping there exists a neighbourhood $V_1(\mu, \varepsilon)$ of zero in R_Δ such that

$$\|x-y\| \in V_1(\mu, \varepsilon) \Rightarrow \Phi(\|x-y\|) \in U_{\mu,\varepsilon}.$$

Suppose now that $V_2(\mu, \varepsilon)$ is a circled neighbourhood of zero in X such that

$$x-y \in V_2(\mu, \varepsilon) \Rightarrow \|x-y\| \in V_1(\mu, \varepsilon).$$

Since X is a Hausdorff topological vector space it is also a Hausdorff uniform space and let d be a pseudometric on X and $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow x-y \in V_2(\mu, \varepsilon).$$

We shall use the notation

$$V_x(d, \delta) = \{y : y \in X, d(x, y) < \delta\} \quad (\delta > 0).$$

Since the set K is compact there exists a finite set $\{x_1, x_2, \ldots, x_m\} \subset K$ such that for every $x \in K$ there exists $i \in \{1, 2, \ldots, m\}$ such that

$$x \in V_{x_i}(d, \delta).$$

So if we define the functions f_i: $K \to R^+$ $(i = 1, 2, \ldots, m)$ so that

$$f_i(x) = \max\{0, \delta - d(x, x_i)\}$$

for every $x \in K$ and $i \in \{1, 2, \ldots, m\}$ it follows that

$$f_i(x) \neq 0 \Leftrightarrow d(x, x_i) < \delta.$$

Since for every $x \in K$ there exists at least one $i \in \{1, 2, \ldots, m\}$ such that $f_i(x) \neq 0$ we conclude that for every $x \in K$,

$$s(x) = \sum_{i=1}^{m} f_i(x) \neq 0$$

and that all mappings f_i $(i=1, 2, \ldots, m)$ are continuous since the mapping $x \mapsto d(x, x_i)$ is continuous for every $i \in \{1, 2, \ldots, m\}$. Now, let

$$h(x) = \frac{1}{s(x)} \sum_{i=1}^{m} f_i(x) x_i \quad \text{for all} \quad x \in K.$$

Then $h(K) \subset \text{co}\, K$ and h is a continuous mapping from K into a finite dimensional subspace of X. Further we have

$$\|hx - x\| = \left\| \frac{1}{s(x)} \sum_{i=1}^{m} f_i(x) x_i - \frac{1}{s(x)} \sum_{i=1}^{m} f_i(x) x \right\| =$$

$$= \left\| \frac{1}{s(x)} f_i(x)(x - x_i) \right\| \leqq \frac{1}{s(x)} \sum_{i=1}^{m} f_i(x) \Phi(\|x - x_i\|).$$

Since $f_i(x) \neq 0 \Leftrightarrow d(x, x_i) < \delta$ it follows that for every $x \in K$ such that $f_i(x) \neq 0$ we have that

$$\Phi(\|x - x_i\|) \in U_{\mu, \varepsilon}$$

and so

$$\|hx - x\|(t) \leqq \frac{1}{s(x)} \sum_{i=1}^{m} f_i(x) \Phi(\|x - x_i\|)(t) < \frac{1}{s(x)} \sum_{i=1}^{m} f_i(x) \varepsilon = \varepsilon \quad \text{for every} \quad t \in \mu.$$

So we have $\|hx - x\| \in U_{\mu, \varepsilon}$, which implies $hx - x \in U$ and the proof is complete.

Corollary 1. *Every convex subset A of type Φ of a Hausdorff topological vector space is admissible.*

Proof. If K is a compact subset of A and U is an arbitrary neighbourhood of zero, the Theorem implies the existence of a finite dimensional continuous mapping $h\colon K \to \text{co}\, K$ with the following property:

$$x - hx \in U \quad \text{for all} \quad x \in K.$$

Since A is convex it follows that $\text{co}\, K \subset A$ and so A is admissible.

Corollary 2. *Every Hausdorff topological vector space of type Φ is admissible.*

Corollary 3. *Let A be a closed and convex subset of type Φ of a Hausdorff topological vector space E and $h\colon A \to A$ be a continuous mapping such that $\overline{h(A)}$ is compact. Then there exists at least one fixed point of the mapping h.*

Proof. Since A is admissible we can apply a fixed point theorem from [3] and so the set of fixed points of the mapping h is nonempty.

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DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
UNIVERSITY OF NOVI SAD
ULICA DR ILIJE DJURIČIĆA 4
21000 NOVI SAD
YUGOSLAVIA